On the Laplace transform of the lognormal distribution

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Abstract

Integral transforms of the lognormal distribution are of great importance in statistics and probability, yet closed-form expressions do not exist. A wide variety of methods have been employed to provide approximations, both analytical and numerical. In this paper, we analyse a closed-form approximation $\widetilde{\mathcal{L}}(\theta)$ of the Laplace transform $\mathcal{L}(\theta)$ which is obtained via a modified version of Laplace's method. This approximation, given in terms of the Lambert $\mathscr{W}(\cdot)$ function, is tractable enough for applications. We prove that $\widetilde{\mathcal{L}}(\theta)$ is asymptotically equivalent to $\mathcal{L}(\theta)$ as $\theta \to \infty$. We apply this result to construct a reliable Monte Carlo estimator of $\mathcal{L}(\theta)$ and prove it to be logarithmically efficient in the rare event sense as $\theta \to \infty$.

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1 Introduction

The lognormal distribution is of major importance in probability and statistics as it arises naturally in a wide variety of applications. For instance, the central limit theorem implies that the limit distribution of a product of random variables often can be well approximated by the lognormal distribution. Hence it is not surprising that the lognormal distribution is frequently employed in disciplines such as engineering, economics, insurance or finance, and it often appears in modeling across the sciences including chemistry, physics, biology, physiology, ecology, environmental sciences and geology; even social sciences and linguistics, see [2, 10, 12, 18, 20].

A random variable X has a lognormal distribution if $X = \mathrm{e}^Y$ where Y is a $\mathrm{N}(\mu, \sigma^2)$ random variable. The density of X is

$$f(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(\log x - \mu)^2}{2\sigma^2} \right\}, \qquad x \in \mathbb{R}^+.$$

The Laplace transform of X is

$$\mathbb{E}\{\exp(-\theta \mathrm{e}^{Y})\} = \mathrm{e}^{-\theta\mu}\mathbb{E}\{\exp(-\theta \mathrm{e}^{Y_0})\}, \quad Y_0 \sim \mathrm{N}(0, \sigma^2), \tag{1.1}$$

and because of this relation we need only consider $\mathcal{L}(\theta) = \mathbb{E}\{\exp(-\theta \mathrm{e}^{Y_0})\}$. The defining integral of the Laplace transform

$$\begin{aligned} \mathcal{L}(\theta) &= \mathbb{E}\{\exp(-\theta \mathrm{e}^{Y_0})\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big\{-\theta \mathrm{e}^{y} - \frac{1}{2\sigma^2}y^2\Big\}\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\big\{-h_\theta(y)\big\}\mathrm{d}y, \qquad h_\theta(y) = \theta \mathrm{e}^{y} + \frac{1}{2\sigma^2}y^2, \end{aligned} \tag{1.2}$$

has no closed form expression. For a complex argument θ the integral defining $\mathcal{L}(\theta)$ diverges when $\Re(\theta) < 0$ and in consequence, the function $\mathcal{L}(\theta)$ is not defined in the left half of the complex plane and fails to be analytic on the imaginary axis. In the absence of a closed-form expression it is desirable to have sharp approximations for the transforms of the lognormal distributions as this paves the way for obtaining the distribution of a sum of i.i.d. lognormal random variables via transform inversion.

In this paper we analyze the closed form approximation $\widetilde{\mathcal{L}}(\theta)$, $\theta > 0$, of the Laplace transform of the lognormal distribution which we reported in [21] and was obtained via a modified version of Laplace's method. The approximation is

$$\widetilde{\mathcal{L}}(\theta) = \frac{1}{\sqrt{1+\mathcal{W}(\theta\sigma^2)}} \exp\left\{ -\frac{1}{2\sigma^2}\mathcal{W}(\theta\sigma^2)^2 - \frac{1}{\sigma^2}\mathcal{W}(\theta\sigma^2)\right\}, \qquad \theta \in \mathbb{R}^+. \tag{1.3}$$

In this expression, $\mathcal{W}(\cdot)$ is the *Lambert W function* which is defined as the solution of the equation $\mathcal{W}(x)\mathrm{e}^{\mathcal{W}(x)} = x$. This function has been widely studied in the last 20 years mainly due to the advent of fast computational methods, cf. [9]. Roughly speaking, the standard Laplace method [11] states that

$$\int_a^b \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\lambda h(y)}\, g(y)\,\mathrm{d}y = \frac{\mathrm{e}^{-\lambda h(\rho)}}{\sqrt{\lambda h''(\rho)}}\, g(\rho)\,\big(1 + O(\lambda^{-1})\big), \qquad \lambda \to \infty, \tag{1.4}$$

where $g(t)$ and $h(t)$ are functions such that $g(t)$ is "well behaved" and $h(t)$ has a unique global minimum at $\rho \in (a, b)$. Therefore, the leading term of the expression on the right hand side of (1.4) can be used as an asymptotic approximation of the integral on the left hand side. Surprisingly, this approximation can be very accurate for very general functions $g(t)$ and $h(t)$, not only in the asymptotic setting, but also for relatively small values of λ. In our case, we take $g(t) \equiv 1$ and identify $\lambda h(y)$ with $h_\theta(y)$ in the integral from (1.2). Intuitively, the role of λ is given by the second derivative of $h_\theta(y)$ at the minimum of $h_\theta(y)$, and we use the standard Laplace method for a situation where the function $h(y)$ depends on λ also. The essential features of $h_\theta(y)$ for this approach are as follows.

Lemma 1.1. *Let $\theta > 0$. Then the exponent $h_\theta(y)$ from (1.2) is convex, attains its minimum value at $-\mathcal{W}(\theta\sigma^2)$, and the second derivative of $h_\theta(y)$ at the minimum is $1/\sigma(\theta)^2$ with $\sigma(\theta)^2 = \sigma^2/\{\mathcal{W}(\theta\sigma^2) + 1\}$.*

To understand the error of the approximation (1.3) we note the general result

$$\mathcal{W}(x) \sim \log(x), \quad x \to \infty.$$

Using this we see that $\mathcal{W}(\theta\sigma^2) \sim \log(\theta)$ and $\sigma(\theta)^2 \sim 1/\log(\theta)$ as $\theta \to \infty$ and this leads to the approximation

$$\mathcal{L}(\theta) \;=\; \widetilde{\mathcal{L}}(\theta)\big(1 + O(\log^{-1}(\theta))\big), \qquad \theta \to \infty.$$

In this paper we prove this approximation and obtain an expansion of the error term. Moreover, we will obtain a probabilistic representation of the error term and construct a Monte Carlo estimator for the Laplace transform. We remark that one must be careful in the implementation of Monte Carlo estimators as naive simulation can lead to unreliable approximations. We show that our proposal corresponds to an importance sampling estimator which delivers reliable estimates for any value of the argument by proving its asymptotic efficiency in a rare-event sense to be defined. We further provide numerical comparisons of our proposal against other methods available in the literature.

We note that the problem of approximating the transforms of a lognormal is notoriously problematic and has been long standing. Therefore, a significant number of methods have been developed to approximate both the Laplace transform and the characteristic functions of the lognormal distribution. We give a more complete literature survey at the end in Section 5 and mention here the work which is most relevant for the present paper. Barakat [5] expanded the characteristic function $\varphi(\omega) = \mathcal{L}(-\mathrm{i}\omega)$ by making a series expansion of $\mathrm{e}^{\mathrm{i}\omega(\mathrm{e}^t - t - 1)}$ and integrating terms by terms. Holgate [16] also considered the characteristic function and employed the classical *saddle point method* [11, 23], which consists in applying Cauchy's theorem to deform the path of integration in such a way that it traverses through a saddlepoint of the integrand in the steepest descent direction. As we mention at the end of Section 2, this is similar to the approximation we develop in this paper for the Laplace transform.

Gubner [14] employed (as many others) numerical integration techniques and was the first in proposing alternative path contours to reduce the oscillatory behavior of the integrand. This approach was further extended in Tellambura and Seranarte [24] where specific contours passing through the saddlepoint at a steepest descent rate were proposed; this choice has the effect that oscillations are removed in a neighborhood around the saddlepoint. In addition, [24] also addresses the heavy-tailed nature of the lognormal density by proposing a transformation which delivers an integrand with lighter tails.

In this paper we follow a path somewhat different (although related to the saddlepoint method) from Holgate to approximate the Laplace transform (1.2) of a lognormal distribution by using a variant of the *Laplace method*. To the best of our knowledge, the resulting closed form approximation (1.3) derived from this methodology was first reported in [21].

The paper is organized as follows: in Section 2 we compute the approximation of the Laplace transform of the lognormal distribution and analyze its asymptotic properties; in addition, we extend this result to the complex plane via the saddlepoint method and establish the relationships with the results of Holgate. In Section

3 we construct an importance sampling estimator for approximating the Laplace transform and prove its efficiency properties; we discuss the disadvantages of using naïve Monte Carlo for estimating this Laplace transform. We verify the sharpness of our approximations and present some numerical comparisons in the analysis presented in Section 4. Finally, a more complete literature survey and a summary of the paper are in Section 5.

2 Approximating the lognormal Laplace transform

In this section we first derive a representation of the Laplace transform $\mathcal{L}(\theta)$ and then use this to derive an expansion of $\mathcal{L}(\theta)$. For completeness we start by proving Lemma 1.1.

Proof of Lemma 1.1. The first and second derivatives of $h_\theta(y)$ are

$$h'_\theta(y) = \theta \mathrm{e}^y + \frac{y}{\sigma^2}, \quad h''_\theta(y) = \theta \mathrm{e}^y + \frac{1}{\sigma^2}.$$

Clearly, the second derivative is positive, implying the convexity of $h_\theta(y)$. Equating the first derivative to zero we get $-y\mathrm{e}^{-y} = \theta\sigma^2$ or $y = -\mathcal{W}(\theta\sigma^2)$. Finally, the second derivative at the minimum is

$$\frac{1}{\sigma^2}\{\theta\sigma^2\mathrm{e}^y + 1\} \;=\; \frac{1}{\sigma^2}\{-y+1\} \;=\; \frac{1}{\sigma^2}\{\mathcal{W}(\theta\sigma^2)+1\},$$

as stated in the lemma. □

To shorten the notation slightly, define $w(\theta) = \mathcal{W}(\theta\sigma^2)$ and recall the notation $\sigma(\theta)^2 = \sigma^2/\{w(\theta)+1\}$. Furthermore, define

$$I(\theta) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{w(\theta)}{\sigma^2}\big(\mathrm{e}^{z\sigma(\theta)} - 1 - z\sigma(\theta)\big) - \frac{\sigma(\theta)^2}{2\sigma^2}z^2 \right\}\mathrm{d}z. \tag{2.1}$$

Recalling the expression (1.3) for $\widetilde{\mathcal{L}}(\theta)$, we then have the following representation result.

Proposition 2.1. *The Laplace transform $\mathcal{L}(\theta)$ can be written as*

$$\mathcal{L}(\theta) = \widetilde{\mathcal{L}}(\theta) I(\theta). \tag{2.2}$$

Furthermore,

$$I(\theta) \;=\; \mathbb{E}g\big(\sigma(\theta)U;\theta\big) \;=\; \sqrt{1+w(\theta)}\,\mathbb{E}\vartheta(\sigma U;\theta), \tag{2.3}$$

where U is standard normal random variable, and

$$g(y;\theta) = \exp\left\{ -\frac{w(\theta)}{\sigma^2}\Big(\mathrm{e}^y - 1 - y - \frac{y^2}{2}\Big)\right\}, \qquad \vartheta(y;\theta) = \exp\left\{ -\frac{w(\theta)}{\sigma^2}\big(\mathrm{e}^y - 1 - y\big)\right\}.$$

Proof. We first note that

$$h_\theta\big(-w(\theta)\big) \;=\; \frac{1}{\sigma^2}\theta\sigma^2 \mathrm{e}^{-w(\theta)} + \frac{w(\theta)^2}{2\sigma^2} \;=\; \frac{w(\theta)}{\sigma^2} + \frac{w(\theta)^2}{2\sigma^2}.$$

Using this and a change of variable $z \to -w(\theta) + \sigma(\theta)z$, we rewrite the defining integral in (1.2) as

$$\begin{aligned}\mathcal{L}(\theta) \;&=\; \frac{\sigma(\theta)}{\sigma}\exp\big\{-h_\theta\big(-w(\theta)\big)\big\}\int_{-\infty}^{\infty}\frac{\exp\big\{-k_\theta(z)\big\}}{\sqrt{2\pi}}\,\mathrm{d}z\\ &=\; \widetilde{\mathcal{L}}(\theta)I(\theta),\end{aligned}$$

where $k_\theta(z) = h_\theta\big(-w(\theta)+\sigma(\theta)z\big) - h_\theta\big(-w(\theta)\big)$. Since

$$\begin{aligned}k_\theta(z) \;&=\; \frac{w(\theta)\mathrm{e}^{\sigma(\theta)z}}{\sigma^2} + \frac{[-w(\theta)+\sigma(\theta)z]^2}{2\sigma^2} - \frac{w(\theta)}{\sigma^2} - \frac{w(\theta)^2}{2\sigma^2}\\ &=\; \frac{w(\theta)}{\sigma^2}\big(\mathrm{e}^{\sigma(\theta)z} - 1 - z\sigma(\theta)\big) + \frac{\sigma(\theta)^2}{2\sigma^2}z^2\,,\end{aligned}$$

we have clearly

$$I(\theta) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\vartheta\big(\sigma(\theta)z;\theta\big)\exp\Big\{-\frac{\sigma(\theta)^2}{2\sigma^2}z^2\Big\}\,\mathrm{d}z,$$

and making the change of variable $u = \sigma(\theta)z/\sigma$ gives $I(\theta) = \sqrt{1+w(\theta)}\mathbb{E}\vartheta(\sigma U;\theta)$. Also, we have

$$\begin{aligned}I(\theta) \;&=\; \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}g\big(\sigma(\theta)z;\theta\big)\exp\Big\{-(w(\theta)+1)\frac{\sigma(\theta)^2}{2\sigma^2}z^2\Big\}\,\mathrm{d}z\\ &=\; \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}g\big(\sigma(\theta)z;\theta\big)\exp\Big\{-\frac{1}{2}z^2\Big\},\end{aligned}$$

which gives the representation $I(\theta) = \mathbb{E}g\big(\sigma(\theta)U;\theta\big)$. □

The representation of the integral $I(\theta)$ as an expected value has a crucial advantage since it allows us to evaluate $I(\theta)$ via careful chosen Monte Carlo methods (section 3). Observe that the function $g(\,\cdot\,;\theta)$ roughly equals 1 in a neighborhood of 0; in consequence, the value $\mathbb{E}\big[g(\sigma(\theta)\,U;\theta)\big]$ is relatively close to 1.

In the following, we will expand the Laplace transform $\mathcal{L}(\theta)$ by using an asymptotic series representation of $I(\theta)$. The argument has two parts: looking at an inner region and an outer region in the integral. In the inner part we use a Taylor expansion, and the outer part is bounded using the convexity of $h_\theta(y)$. In fact, we will prove that the modified Laplace method delivers an approximation with an error which is asymptotically negligible. Moreover, it turns out that this approximation remains accurate in all the domain of convergence of θ, as we will empirically corroborate in the numerical examples in Section 4.

Proposition 2.2. *As $\theta \to \infty$, we have*

$$\mathcal{L}(\theta) \;=\; \widetilde{\mathcal{L}}(\theta)I(\theta) \;=\; \widetilde{\mathcal{L}}(\theta)\big(1+O\big(\log(\theta)^{-1}\big)\big). \tag{2.4}$$

Furthermore the error term can be expanded as

$$I(\theta) = \begin{cases} 1 & \theta = 0, \\ 1 - \dfrac{3w(\theta)\sigma(\theta)^4}{24\sigma^2} + \dfrac{5w(\theta)^2\sigma(\theta)^6}{24\sigma^4} + o\big(\sigma(\theta)^2\big) & \theta \to \infty. \end{cases} \tag{2.5}$$

Proof. Expansion of the exponential part $-k_\theta(z) = -[h_\theta(-w(\theta)+\sigma(\theta)z) - h_\theta(-w(\theta)]$ of the integrand in $I(\theta)$ gives

$$-\frac{1}{2}z^2 - \frac{w(\theta)\sigma(\theta)^3}{6\sigma^2}z^3 - \frac{w(\theta)\sigma(\theta)^4}{24\sigma^2}z^4 + O\big(\sigma(\theta)^3|z|^5\big)\,.$$

Expanding the exponential of the last three terms here we find that the integrand in $I(\theta)$ is

$$\frac{\exp(-\frac{1}{2}z^2)}{\sqrt{2\pi}}\Big\{1 - \frac{w(\theta)\sigma(\theta)^3}{6\sigma^2}z^3 - \frac{w(\theta)\sigma(\theta)^4}{24\sigma^2}z^4 + \frac{w(\theta)^2\sigma(\theta)^6}{72\sigma^2}z^6 + O(\sigma(\theta)^3(|z|^5 + |z|^9))\Big\}. \tag{2.6}$$

We use this expansion for $|z| < \sigma(\theta)^{-\epsilon}$, where ϵ is a small positive number with $9\epsilon < 1$. Integrating (2.6) over this region gives (2.5) since

$$\int_{|z|<\sigma(\theta)^{-\epsilon}} \frac{z^k}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)\,\mathrm{d}z \;=\; \int_{-\infty}^{\infty}\frac{z^k}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}z^2\Big\}\,\mathrm{d}z + O\Big(\exp\Big\{-\frac{1}{4}\sigma(\theta)^{-\epsilon}\Big\}\Big).$$

What is left is to bound the part of the integral in $I(\theta)$ from the region $|z| > \sigma(\theta)^{-\epsilon}$. Due to the convexity of $k_\theta(z)$ we have that the integral over $z > \sigma(\theta)^{-\epsilon}$ is bounded by

$$\frac{1}{\sqrt{2\pi}}\frac{\exp\{-k_\theta(\sigma(\theta)^{-\epsilon})\}}{k'_\theta(\sigma(\theta)^{-\epsilon})} = O\Big(\exp\Big\{-\frac{1}{2}\sigma(\theta)^{-2\epsilon}\Big\}\Big) \;=\; o\big(\sigma(\theta)^2\big),$$

and with a similar bound for the region $z < -\sigma(\theta)^{-\epsilon}$. We have thus proved (2.5), and (2.4) is a consequence of this, remembering that $\sigma(\theta)^2 \sim 1/\log(\theta)$. □

In the following corollary we summarize the asymptotic behavior of $\mathcal{L}(\theta)$ in the rough form of logarithmic asymptotics familiar from large deviations theory:

Corollary 2.3. *We have*

$$\lim_{\theta\to\infty}\mathcal{L}(\theta) = \lim_{\theta\to\infty}\widetilde{\mathcal{L}}(\theta) = 0 \quad \textit{and} \quad \lim_{\theta\to\infty}\frac{\log\mathcal{L}(\theta)}{(\log\theta)^2} \;=\; -\frac{1}{2\sigma^2}.$$

Proof. From the very definition of the Laplace transform we have of course that $\mathcal{L}(\theta)$ tends to zero for $\theta \to \infty$, and (2.4) shows that $\widetilde{\mathcal{L}}(\theta)$ has the same limit. For the second statement we write

$$\log\mathcal{L}(\theta) = \frac{1}{2}\log\big(1 + w(\theta)\big) - \frac{w(\theta)^2}{2\sigma^2} - \frac{w(\theta)}{\sigma^2} + O\big((\log\theta)^{-1}\big) \sim -\frac{(\log\theta)^2}{2\sigma^2}.$$

□

Finally, we discuss the approximation obtained by using the so-called asymptotic saddlepoint methodology [11, 23]. For that purpose we consider the complex function

$$\mathcal{L}(z) = \int_0^\infty \mathrm{e}^{-zx}\,\mathrm{d}F(x) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\Big\{ -z\mathrm{e}^t - \frac{t^2}{2\sigma^2}\Big\}\,\mathrm{d}t, \qquad \Re(z) \geq 0.$$

The saddlepoint method makes use of the Cauchy-Goursat theorem to deform the contour of integration so the new contour traverses the saddlepoint ρ_z of the function

$$h_z(t) \;=\; -z\mathrm{e}^t - t^2/2\sigma^2, \qquad t \in \mathbb{C}.$$

This is possible because $h_z(t)$ is a complex entire function with a unique root $t = \rho_z$ of the equation $h_z'(t) = 0$ which is also a saddlepoint of the functions defining the real and imaginary parts of h_z. Under such circumstances Perron's saddlepoint method indicates that we can select a new contour for which the maximum of $\Re(h_z(t))$ over the contour is reached at the saddlepoint ρ_z, and $\Im(h_z(t))$ is approximately constant over the contour in a neighborhood of the saddlepoint. In consequence, the selected contour is such that the maximum of $|\mathrm{e}^{h_z(t)}|$ is reached at the saddlepoint and most of the contribution to the integral comes from the section of the contour in the neighborhood of the saddlepoint. Thus, the Laplace method can be adapted to provide an approximation of this contour integral. The resulting approximation is the complex analogue of (1.3).

The approximation of the function $\mathcal{L}(z)$ obtained by applying the saddlepoint methodology is [cf. 11, p. 84]

$$\mathcal{L}(z) \;\approx\; \frac{1}{\sqrt{1+\mathcal{W}(z\sigma^2)}} \exp\Big\{-\frac{1}{2\sigma^2}\mathcal{W}(z\sigma^2)^2 - \frac{1}{\sigma^2}\mathcal{W}(z\sigma^2)\Big\}, \qquad \Re(z) > 0. \quad (2.7)$$

This approximation is relevant for the whole domain of convergence of $\mathcal{L}(z)$ in the complex plane. In particular, when restricted to the imaginary axis it coincides with the approximation of the characteristic function given by Holgate [16]. Similarly, when evaluated in the positive reals it coincides with the approximation (1.3) studied in this paper.

3 Efficient Monte Carlo

The approximation of the Laplace transform of the lognormal distribution suggested in the previous section turns out to be reasonably sharp for all positive values of the argument θ when the value of the parameter σ is small; however, the quality of the approximation deteriorates as the value of σ increases (see the numerical results in Section 4 and the form of the remainder terms in (2.5)). When computing transforms it is crucial to come up with approximations which remain sharp for all values of σ and all over the domain of the transform, in particular in the tail regions. Hence it is desirable to be able to achieve errors within certain preselected margins. Resorting to numerical integration methods is a natural choice so various proposals employing this approach have emerged over the years, [7, 14, 24]. However, the difficulty of approximating the defining integral is such that most of the methods proposed are very complicated and deliver unreliable results as corroborated in our numerical examples.

An alternative is the Monte Carlo method. Such an approach has two notable advantages: 1) the approximations can be sharpened at the cost of computational effort; 2) the precision of the estimates can be assessed with accuracy. The basic version, known as *Crude Monte Carlo*, consists in simulating a sequence $X_1,\dots,X_R$ of i.i.d. random variables with common distribution $\mathrm{LN}(0,\sigma^2)$, then applying the transformation $x\mapsto \mathrm{e}^{-\theta x}$ to each random variable and finally returning the arithmetic average of the transformed sequence as an estimator of the lognormal Laplace transform $\mathcal{L}(\theta)$. The Law of Large Numbers ensures unbiasedness of this estimator while the Central Limit Theorem implies that the error, defined for example as the half-width of the confidence interval, can be made arbitrarily small by choosing R large enough. However:

Proposition 3.1. *Let $X\sim \mathrm{LN}(0,\sigma^2)$. Then* $\lim_{\theta\to\infty}\dfrac{\mathrm{Var}(\mathrm{e}^{-\theta X})}{\mathcal{L}(\theta)^2}=\infty$.

Proof. According to Proposition 2.2 we have that

$$\begin{aligned}\lim_{\theta\to\infty}\frac{\mathrm{Var}(\mathrm{e}^{-\theta X})}{\mathcal{L}(\theta)^2}&=\lim_{\theta\to\infty}\frac{\mathcal{L}(2\theta)}{\mathcal{L}(\theta)^2}-1\\&=\lim_{\theta\to\infty}\frac{1+w(\theta)}{\sqrt{1+w(2\theta)}}\frac{\exp\big\{-\frac{1}{2\sigma^2}w(2\theta)^2-\frac{1}{\sigma^2}w(2\theta)\big\}}{\exp\big\{-\frac{2}{2\sigma^2}w(\theta)^2-\frac{2}{\sigma^2}w(\theta)\big\}}-1\\&=\infty,\end{aligned}$$

where in the last step we use that $w(\lambda)\sim\log\lambda$ for $\lambda=\theta$ and for $\lambda=2\theta$, so that $2w(\theta)^2-w(2\theta)\sim(\log\theta)^2$. □

Note that the result is also true for $X\sim\mathrm{LN}(\mu,\sigma^2)$ due to the relation (1.1).

The result implies that the Crude Monte Carlo estimation of $\mathcal{L}(\theta)$ faces the problem of a relative error that goes to infinity so that a huge value of R is required if θ is large. This is exactly the same issue as arising in rare-event simulation ([3, Ch. VI]), where the standard efficiency concepts are the following. A given unbiased estimator $\widehat{\mathcal{L}}(\theta)$ of $\mathcal{L}(\theta)$ is *strongly efficient* or has *bounded relative error* if

$$\limsup_{\theta\to\infty}\frac{\mathrm{Var}\,\widehat{\mathcal{L}}(\theta)}{\mathcal{L}(\theta)^2}<\infty.$$

This efficiency property implies that the number of replications required to estimate $\mathcal{L}(\theta)$ with a certain fixed relative precision remains bounded as $\theta\to\infty$. A weaker criterion is *logarithmic efficiency* defined as

$$\limsup_{\theta\to\infty}\frac{\mathrm{Var}\,\widehat{\mathcal{L}}(\theta)}{\mathcal{L}(\theta)^{2-\epsilon}}=0,\qquad\forall\epsilon>0.$$

This criterion implies that the number of replications needed for achieving certain relative precision grows at most at rate of order $|\log(\mathcal{L}(\theta))|$. While bounded relative error is clearly a stronger form of efficiency, it is widely accepted that for practical purposes (numerical implementations), there is no substantial difference between these two criteria, although it is often more involved to prove bounded relative error than logarithmic efficiency.

Our objective is to use variance reduction to construct an efficient estimator of the lognormal Laplace transform $\mathcal{L}(\theta)$. For that purpose we will construct a new estimator employing the probabilistic representation of $\mathcal{L}(\theta)$ obtained in Proposition 2.1. More specifically, we use *Importance Sampling* which consists in sampling from an alternative distribution and then modifying by the likelihood ratio to remove the bias. In general, this method requires a careful analysis in order to be effective as it not always produces a reduction in variance. We proceed to discuss these ideas in detail.

Recall Proposition 2.1 which says that for any $\theta > 0$, it holds that

$$\mathcal{L}(\theta) \;=\; \widetilde{\mathcal{L}}(\theta)\mathbb{E}\big[g(\sigma(\theta)\,U;\theta)\big]$$

where $\widetilde{\mathcal{L}}(\theta)$ is the approximation (1.3) of the Laplace transform, U is a normal standard random variable and the function g is defined in Proposition 2.1. A naïve approach is to use a Crude Monte Carlo estimator of $\mathbb{E}[g(\sigma(\theta)\,U;\theta)]$, i.e. simulate $U \sim \mathrm{N}\,(0,1)$ and return $g(\sigma(\theta)\,U;\theta)$. We refer to this estimator as *Naïve Monte Carlo* and denote it $\widehat{\mathcal{L}}_N(\theta)$ in order to distinguish from the Crude Monte Carlo estimator discussed previously.

The Naïve Monte Carlo estimator $\widehat{\mathcal{L}}_N(\theta)$ is still highly unreliable (in spite of the apparent sharpness observed in the numerical examples in Section 4) as it turns out it has infinite variance when $\theta > \mathrm{e}^1\sigma^{-2}$. For proving this, consider the second moment

$$\begin{aligned}
&\mathbb{E}[g(\sigma(\theta)\,U;\theta)^2] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\Big\{-\frac{2w(\theta)}{\sigma^2}\big(\mathrm{e}^{\sigma(\theta)z}-1-\sigma(\theta)z-\frac{1}{2}\sigma(\theta)^2z^2\big)-\frac{z^2}{2}\Big\}\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\Big\{-\frac{2w(\theta)}{\sigma^2}\big(\mathrm{e}^{\sigma(\theta)z}-1-\sigma(\theta)z\big)+\frac{w(\theta)-1}{w(\theta)+1}\cdot\frac{z^2}{2}\Big\}\mathrm{d}z. \qquad (3.1)
\end{aligned}$$

If $z \to -\infty$, the exponential term $\mathrm{e}^{\sigma(\theta)z}$ vanishes and we are left with a second order polynomial with leading coefficient $[w(\theta)-1]/[2(w(\theta)+1)]$. This coefficient is positive if $w(\theta) > 1$ which occurs if and only if $\theta > \mathrm{e}^1/\sigma^2$. In such case the integrand goes to infinity as $t \to -\infty$.

The argument just given shows that although the random variable $g(\sigma(\theta)\,U;\theta)$ has the correct expected value, the naïve Monte Carlo estimator is bound to deliver unreliable estimates because of an infinite variance. This observation is not entirely surprising as exponential transformations of light tailed random variables (as the one obtained by applying the function g^2 to a normal random variable) often yields heavy-tailed distributions with infinite moments. To fix this problem we propose a second estimator which is based on the alternative representation in Proposition 2.1. Recall that

$$\mathcal{L}(\theta) \;=\; \widetilde{\mathcal{L}}(\theta)\sqrt{1+w(\theta)}\mathbb{E}\big[\vartheta(\sigma U;\theta)\big], \quad \vartheta(t;\theta) \;=\; \exp\Big\{-\frac{w(\theta)}{\sigma^2}\big(\mathrm{e}^t-1-t\big)\Big\}.$$

where $\sigma U \sim \mathrm{N}(0, \sigma^2)$. Hence, if $Y \sim \mathrm{N}(0, \sigma^2)$ then

$$\widehat{\mathcal{L}}_{IS}(\theta) = \exp\Big\{-\frac{w(\theta)^2}{2\sigma^2} - \frac{w(\theta)}{\sigma^2}\Big\} \cdot \vartheta(Y, \theta). \tag{3.2}$$

is an unbiased estimator of $\mathcal{L}(\theta)$. The notation is motivated by $\widehat{\mathcal{L}}_{IS}(\theta)$ being derived from the naïve estimator by importance sampling, that is, by $\mathbb{E}g(Y)$ with Y having a $\mathrm{N}\big(0, \sigma^2(\theta)\big)$ density $f_{\sigma^2(\theta)}$ being estimated by $g(Y)f_{\sigma^2(\theta)}(Y)/f_{\sigma^2}(Y)$ where Y now has a $\mathrm{N}\big(0, \sigma^2\big)$ density f_{σ^2}. This follows since

$$\frac{f_{\sigma^2(\theta)}(y)}{f_{\sigma^2}(y)} \;=\; \frac{\sigma}{\sigma(\theta)}\exp\Big\{-\Big(\frac{1}{2\sigma(\theta)^2} - \frac{1}{2\sigma^2}\Big)y^2\Big\} \;=\; \sqrt{1+w(\theta)}\exp\Big\{-\frac{w(\theta)}{2\sigma^2}y^2\Big\},$$

and

$$g(y;\theta)\frac{f_{\sigma^2(\theta)}(y)}{f_{\sigma^2}(y)} = \sqrt{1+w(\theta)}\vartheta(y;\theta)\,.$$

Moreover, in the next proposition we show that $\widehat{\mathcal{L}}_{IS}(\theta)$ achieves logarithmic efficiency as $\theta \to \infty$.

Proposition 3.2. *The estimator $\widehat{\mathcal{L}}_{IS}(\theta)$ is an unbiased estimator of the Laplace transform of the lognormal distribution* $\mathrm{LN}(0, \sigma^2)$. *For $\theta \to \infty$ and $\frac{\sigma}{\log\theta} \to 0$ we have $\mathcal{L}(\theta) \to 0$ and the IS estimator is logarithmic efficient. For $\theta \to \infty$ and $\frac{\sigma}{\log\theta}$ bounded away from zero the Laplace transform $\mathcal{L}(\theta)$ is bounded away from zero as well, and the IS estimator has bounded relative error.*

Proof. The unbiasedness has already been discussed above. We write $\widehat{\mathcal{L}}_{IS}(\theta) = L_0(\theta)\vartheta(Y;\theta)$, with $L_0(\theta) = \exp\{-[w(\theta)^2 + 2w(\theta)]/(2\sigma^2)\}$, and $\mathcal{L}(\theta) = L_0(\theta)\frac{\sigma(\theta)}{\sigma}I(\theta)$. Also we note that $0 \le \vartheta(t;\theta) \le 1$ so that $\mathrm{Var}[\widehat{\mathcal{L}}_{IS}(\theta)] \le L_0(\theta)^2$.

When $\sigma(\theta) \to 0$ we have from Proposition 2.2 that $I(\theta) \to 1$. More generally, from the proof of Proposition 2.2 we have that for $\sigma(\theta) \le c$, for some constant c, there exist $0 < c_1 < c_2 < \infty$ so that $c_1 < I(\theta) < c_2$. In this case we get

$$\begin{aligned}\lim \frac{\mathrm{Var}[\widehat{\mathcal{L}}_{IS}(\theta)]}{\mathcal{L}(\theta)^{2-\epsilon}} &\le \lim c_1^{-(2-\epsilon)} L_0(\theta)^{\epsilon}\big(1+w(\theta)\big)^{1-\epsilon/2} \\ &= \lim c_1^{-(2-\epsilon)} \exp\{-\epsilon[w(\theta)^2 + 2w(\theta)]/(2\sigma^2)\}\big(1+w(\theta)\big)^{1-\epsilon/2}. \end{aligned} \tag{3.3}$$

It is clear that $w(\theta)$ is increasing in σ^2 and calculating the derivative of $[w(\theta)^2 + 2w(\theta)]/(2\sigma^2)$ it is seen that this term is decreasing in σ^2. The worst case scenario in (3.3) is therefore for the largest value of σ^2, which is of order $c^2\log(\theta)$ under the restriction $\sigma(\theta) \le c$, and in this case $w(\theta) \sim \log(\theta\sigma^2) \to \infty$. The limit in (3.3) then becomes

$$\lim \exp\{-\epsilon[w(\theta)^2 + 2w(\theta)]/(2c^2(1+w(\theta)))\}\big(1+w(\theta)\big)^{1-\epsilon/2} = 0.$$

When $\sigma(\theta)$ is no longer bounded we need a different scale to standardize the integral $I(\theta)$. As explained in [4] the relevant scale is $\tau(\theta) = \sqrt{w(\theta)^2 + 2w(\theta) + \sigma^2} - w(\theta)$ so that now

$$\mathcal{L}(\theta) = L_0(\theta)\frac{\tau(\theta)}{\sigma}\tilde{I}(\theta)$$

with $c_1 < \tilde{I}(\theta) < c_2$. When $\sigma(\theta)$ is bounded away from zero we have that $\tau(\theta)/\sigma$ is bounded from below by a constant times $\sigma/w(\theta)$. Instead of (3.3) we get the upper limit

$$\lim \exp\{-\epsilon[w(\theta)^2]/(2\sigma^2)\}\big(w(\theta)/\sigma\big)^{2-\epsilon} = 0,$$

under the assumtion that $\frac{\sigma}{\log\theta} \to 0$.

Finally, when $\frac{\sigma}{\log\theta}$ is bounded away from zero $\tau(\theta)/\sigma$ and $L_0(\theta)$ are both bounded away from zero, implying that $\mathcal{L}(\theta)$ is bounded away from zero. This shows that the IS estimator has bounded relative error in this case. □

In summary we propose the following

Algorithm for generating a single replicate of the IS estimator

1. Simulate $Y \sim \mathrm{N}\,(0, \sigma^2)$.
2. Compute $\vartheta(Y;\theta) = \exp\Big\{-\dfrac{\mathcal{W}(\theta\sigma^2)}{\sigma^2}\big(\mathrm{e}^Y - 1 - Y\big)\Big\}$.
3. Return $\widehat{\mathcal{L}}_{IS}(\theta) = \exp\Big\{-\dfrac{\mathcal{W}^2(\theta\sigma^2) + 2\,\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\Big\}\,\vartheta(Y;\theta)$.

4 Numerical examples

In this section we investigate the numerical performance of our proposals and compare them against several approximations/numerical schemes available in the literature; we further discuss the quality of the approximations. First we investigate the approximations of the Laplace transform. Secondly, we corroborate empirically the efficiency properties of the Monte Carlo estimators considered in this paper.

4.1 Approximations of the Laplace transform

First, we compute our closed form approximation $\widetilde{\mathcal{L}}(\theta)$ given in (1.3). Next, we implement our Importance Sampling Monte Carlo estimator $\widehat{\mathcal{L}}_{\mathrm{IS}}(\theta)$ given in formula (3.2). In our implementation we used 10^8 replications; each estimate is accompanied with a 95% confidence interval.

For comparison purposes we consider the approximation proposed by Barakat [5], and the numerical schemes of Gubner [14] and Tellambura and Seranarte [24]. The scheme of Tellambura and Seranarte [24] is designed to handle both the characteristic function and the Laplace transform. Contrary to this, the scheme of Gubner [14] is designed for the characteristic function only. When using the same scheme for the Laplace transform, we find that the numerical scheme of Tellambura/Seranarte is always superior. For this reason we exclude the results of Gubner in the tables below. Also the approximation of Barakat [5] is designed for the characterisitic function and uses a truncated series expansion. Formally, when using the same expansion for the Laplace transform, we find that the approximation works well for small values of σ and that the series seems not to converge for σ large (roughly σ above 0.5). We have therefore not included this approximation in the second and third tables below. As we will see below the scheme of Tellambura/Seranarte works well except for small values of σ. Because of this we have included in the tables the numerical scheme of Asmussen, Jensen and Rojas-Nandayapa [4] that is designed to work for all values of θ and σ. The scheme is described in Appendix 5. Each approximation presented below is accompanied with its relative error with respect to the numerical scheme described in Appendix 5. The relative error is defined as

$$\mathrm{RE} = \frac{\text{Approximation} - \text{Appendix}}{\text{Appendix}},$$

where Appendix is the result of the numerical scheme described in Appendix 5. When the relative error is very small only the relative error is shown in the tables, otherwise both the approximation and the relative error (in parenthesis) are shown.

We remark that the numerical output corresponding to the numerical schemes of Gubner and Tellambura/Seranarte presented here is obtained by using the Matlab codes provided by the authors. Also, for the series expansion of Barakat [5] we use nine terms as done by Barakat.

Example 1. For our first example we consider $\sigma = 0.0625$. The results are in Table 1.

Table 1: Approximations of $\mathcal{L}(\theta)$ with $\sigma = 0.0625$.

θ	$\widehat{\mathcal{L}}_{\mathrm{IS}}(\theta)$	Appendix	RE $\widetilde{\mathcal{L}}(\theta)$	T/S	RE Barakat
0.0	1.000000	1.000000	0.0e-00	0.945115 (-0.055)	0.0e-00
0.5	0.606235 ± 1.64e-07	0.606235	9.5e-07	0.572894 (-0.055)	-3.5e-12
1.0	0.367880 ± 1.98e-07	0.367880	1.9e-06	0.347607 (-0.055)	-1.2e-10
2.0	0.135862 ± 1.45e-07	0.135862	3.7e-06	0.128344 (-0.055)	1.9e-10
4.0	0.018744 ± 3.95e-08	0.018744	7.1e-06	0.017698 (-0.056)	-3.3e-10
8.0	0.000373 ± 1.52e-09	0.000373	1.3e-05	0.000352 (-0.057)	-5.8e-09

The IS estimator is exact when $\theta = 0$. We see here that for small values of σ the numerical scheme of Tellambura/Seranarte fails in producing a negligible relative error. The approximation of Barakat appears to be the most accurate for small values of σ. Our Laplace Method approximation also delivers very sharp results.

Example 2. Next we consider $\sigma = 1$. The results are in Table 2.

Table 2: Approximations of $\mathcal{L}(\theta)$ with $\sigma = 1$.

θ	$\widehat{\mathcal{L}}_{\mathrm{IS}}(\theta)$	Appendix	$\widetilde{\mathcal{L}}(\theta)$	RE T/S
0.0	1.000000	1.000000	1.000000 (0.0e-00)	-1.2e-14
0.5	0.561717 ± 2.53e-05	0.561707	0.568766 (1.3e-02)	-2.2e-15
1.0	0.381752 ± 2.26e-05	0.381756	0.385738 (1.0e-02)	-1.0e-15
2.0	0.216304 ± 1.60e-05	0.216309	0.217758 (6.7e-03)	-5.7e-14
4.0	0.098042 ± 8.71e-06	0.098051	0.098323 (2.8e-03)	-1.5e-14
8.0	0.034267 ± 3.53e-06	0.034269	0.034252 (-4.9e-04)	-1.0e-12

As already mentioned, in this example the series of Barakat [5] is not converging and therefore not included in the table. The numerical scheme of Tellambura/Seranarte seems to be the one delivering the best results. Our Laplace method approximation (1.3) delivers reliable results (relative error of the order 1% or less) with a relative error diminishing as θ goes to infinity in accordance with the asymptotic equivalence of the approximation (cf. Corollary 2.3).

Example 3. In the third example we consider $\sigma = 4$. The results are shown in Table 3.

Table 3: Approximated Values of $\mathcal{L}(\theta)$ with $\sigma = 4$.

θ	$\widehat{\mathcal{L}}_{\mathrm{IS}}(\theta)$	Appendix	$\widetilde{\mathcal{L}}(\theta)$	RE T/S
0.0	1.000000	1.000000	1.000000 (0.0e-00)	-2.0e-15
0.5	0.513090 $\pm$ 5.95e-05	0.513045	0.516946 (7.63-03)	-1.0e-13
1.0	0.447220 $\pm$ 5.52e-05	0.447199	0.441245 (-1.3e-02)	-6.5e-14
2.0	0.382681 $\pm$ 5.00e-05	0.382686	0.371296 (-3.0e-02)	-4.3e-14
4.0	0.321186 $\pm$ 4.43e-05	0.321194	0.307613 (-4.2e-02)	-2.8e-14
8.0	0.264176 $\pm$ 3.84e-05	0.264181	0.250554 (-5.2e-02)	-1.7e-14

As can be seen the numerical scheme of Tellambura/Seranarte works well for increasing values of σ, whereas the relative error of the approximation $\widetilde{\mathcal{L}}(\theta)$ increases with σ. For $\sigma = 4$ the maximum relative error is approximately 7%.

Numerical investigations show that the Matlab implementation of the numerical scheme of Tellambura/Seranarte eventually breaks down when σ becomes very large. Similarly, one can show theoretically that the approximation $\widetilde{\mathcal{L}}(\theta)$ does not work well for very large values of σ. The reason for this is that the integrand, that is replaced by a gaussian curve in the Laplace approximation, is far from the gaussian form. Nevertheless, our IS Monte Carlo estimator is logarithmic efficient also for $\sigma \to \infty$. Similarly, the numerical scheme in Appendix 5 works as well in this case.

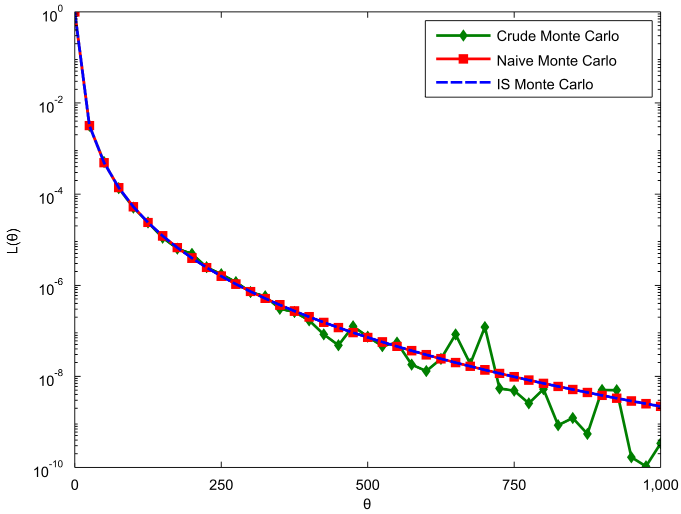
Remark 4.1. We analysed approximations for the characteristic function of the lognormal distribution. Besides the approximation of Barakat, and the numerical schemes of Gubner and Tellambura/Seranarte, we also considered the approximation suggested by Leipnik [19]. Leipnik proposed a series representations of the characteristic function given in terms of Hermite polynomials; however, we could not use this expression to obtain values which could be considered reliable (other authors have faced the same challenges when trying to implement this algorithm [cf. 12]).

The conclusions obtained for the characteristic function were the same as for the Laplace transform so we exclude these numerical examples from our presentation.

4.2 Efficient Monte Carlo

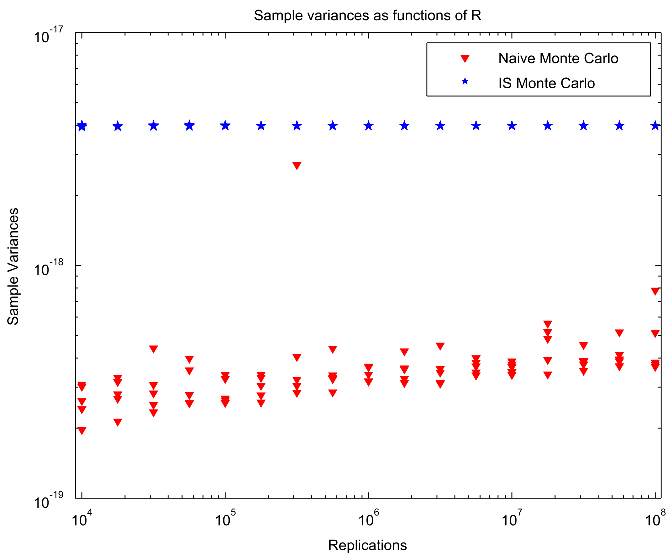
Next we compare the three estimators discussed in this paper: Crude Monte Carlo, Naïve Monte Carlo and Efficient (Importance Sampling) Monte Carlo. For our numerical experiments we took $\sigma = 1$ and used $R = 10^6$ replications. Figure 1 shows the estimates provided by the three methods in logarithmic scale. It is seen that Crude Monte Carlo provides a reasonable approximation of the true Laplace transform for small values of the argument. However, as the value of θ increases, the quality of the crude estimate deteriorates with an increasing relative error (as seen by the jiggled nature of the curve). On the other hand, it appears that the two importance sampling methods discussed in this paper provide sharp approximations as their values are very close to each other (the curves are indistinguishable from each other).

Figure 1: Monte Carlo Estimates for the Laplace transform of the lognormal with $\sigma = 1$.



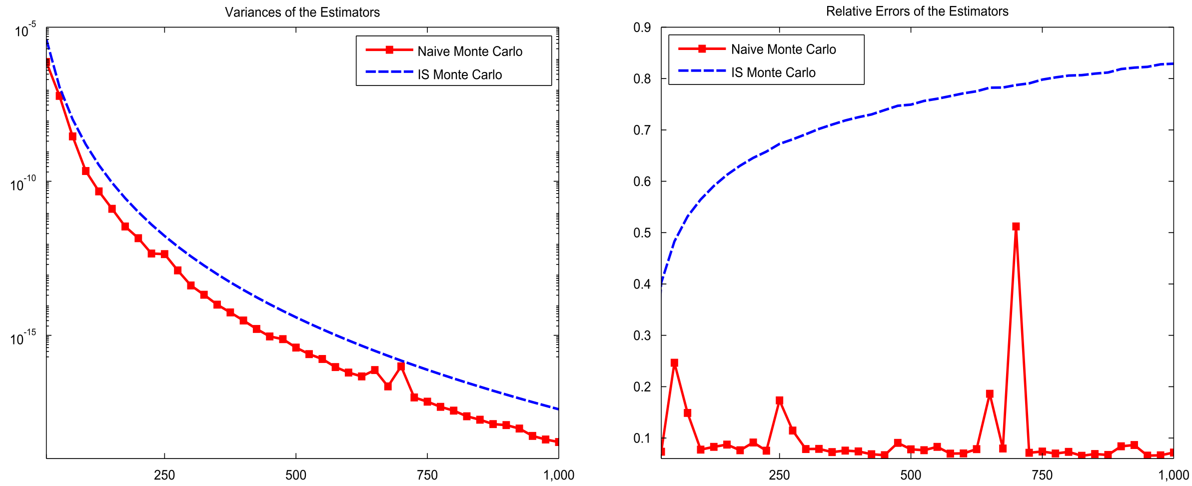
However, one of the two importance sampling estimators (Naïve Monte Carlo) has an infinite variance, so it can provide unreliable estimates. This effect is noted in Figure 2 where the variances of these two estimators are plotted against the number R of replications for 5 runs. The sample variance of the IS estimators is seen to be very stable and practically constant over the 5 runs. In contrast, for the Naïve Monte Carlo there is no convergence (it is even possible to appreciate an increasing trend) and the variation between runs is considerable. These features are the typical behavior of the sample variance of an estimator with infinite variance.

Figure 2: Variance as function of the number of replications with $\sigma = 1$ and $\theta = 1000$.



The variances of these estimators appears in the left panel of Figure 3. It appears that the first estimator has a lower variance but this is only due to the fact that the variance is underestimated (the random "peaks" are also a clear symptom of this problem). On the other hand, the efficient algorithm has a sharp estimate of the real variance which is reflected in the smoothness of the curve. The relative errors are plotted in the panel on the right of Figure 3 in linear scale. The relative error of the IS estimator increases at a rate which appears to be at least logarithmic, thus corroborating its efficiency. In the case of the Naïve Monte Carlo estimator (infinite variance), the estimators of the relative error are unreliable.

Figure 3: Variance and Relative Errors of the importance sampling estimators.



5 Discussion and Conclusion

More on earlier literature: The use of infinite series representations has been one of the most used approaches to deal with the transforms of the lognormal distribution. One of the most obvious attempts in the study of $\mathcal{L}(\theta)$ is using formal series representations as follows: consider the following integral expression for the transform of the lognormal distribution

$$\mathcal{L}(\theta) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\theta \mathrm{e}^t - \frac{(t-\mu)^2}{2\sigma^2} \right\} \mathrm{d}t, \qquad \Re(\theta) > 0\,,$$

replace the term $\mathrm{e}^{-\theta t}$ with its Taylor series, interchange the integral and sum and perform a term by term integration, thus obtaining a formal series representation with the moments of the lognormal distribution as coefficients. This attempt turns to be invalid as the resulting series diverges. This is not surprising because the procedure described above is equivalent to deriving a Taylor series of the function $\mathcal{L}(\theta)$ around the origin, but as noted before, this function is not analytic in the imaginary axis. This pathology is also related to the well known fact that the lognormal distribution is not uniquely determined by its moment sequence (Heyde, 1963, [15]). Other methods using series representations appear as early as 1976. The manuscript of Barouch and Kaufman [6] provides various approximations in terms of series representations which are valid in specific regions; for instance, a series expansion of the lognormal density is employed to produce a closed-form asymptotic approximation in terms of both the Gamma function and its derivatives. However, none of these expressions can deliver reliable estimates in the whole domain of the characteristic function. The representation of the characteristic function proposed by Barakat has the following closed form

$$\phi(\theta) = \mathrm{e}^{-\mathrm{i}\theta} \mathrm{e}^{\theta^2\sigma^2/2} \sum_{n=0}^{\infty} \frac{(-1)^n (\mathrm{i}\sigma)^n}{n!} \, a_n(i\theta)\, H_n(\sigma\theta), \tag{5.1}$$

where $a_n(\theta)$ is the n-th coefficient in the MacLaurin series representation of $\mathrm{e}^{-\theta(\mathrm{e}^y-1-y)}$ and H_n is the n-th Hermite polynomial (notice that we have employed the *probabilist Hermite* polynomials instead of the *physicist Hermite* polynomials in the definition [cf. 1]). We found that the approximation (5.1) of Barakat [5] is sharp for small values of σ^2, but rapidly deteriorates for large values of σ^2 in regions away from the origin. A similar expression is obtained by Leipnik [19], but instead he shows that the characteristic function satisfies a functional differential equation of the form $\varphi'(\omega) = \mathrm{i}\mathrm{e}^{\mu+\sigma^2/2}\varphi(\mathrm{e}^{\sigma^2}\omega)$ (an observation that appeared previously in [6]). Leipnik employs a method due to de Bruijn to solve this functional differential equation: the solution (given as an integral involving the gamma function) is proven to have an explicit convergent infinite series representation in terms of Hermite polynomials which is of the form

$$\varphi(\theta) = \sqrt{\frac{\pi}{2\sigma^2}}\exp\left\{-\frac{\log^2(\theta+\mathrm{i}\pi/2)}{2\sigma^2}\right\}\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{\sigma^n}\,d_n\,H_n\big(\log(\theta+\mathrm{i}\pi/2)/\sigma\big)$$

where d_n are the coefficients in the MacLaurin series representation of the reciprocal of the Gamma function $\Gamma^{-1}(y+1)$. Leipnik includes a recursive formula for calculating the coefficients d_n in terms of Euler's constant and the Riemann Zeta function; this recursion facilitates the calculation of this series representation. However, the solution of the functional differential equation cannot be extended to the whole complex plane, so it appears that this approximation only applies for the characteristic function (in fact, we were not able to obtain a reliable numerical estimate using any of these formulae).

In his study of the characteristic function, cf. [16], Holgate applied the *Lagrange inversion theorem* to the equivalence $t\mathrm{e}^{-t} = \mathrm{i}\sigma^2\omega$ to obtain an asymptotic series representation of the saddle point function $\rho(\omega)$, which inserted into expression (2.7) provides a representation of the function $\varphi(\omega)$ in terms of an asymptotic infinite series. However, the resulting series oscillates wildly and cannot provide a reliable numerical approximations. Finally, another interesting and somewhat different approach which delivers closed-form expressions is given by Rossberg [22], who provides a representation of general Laplace transforms in terms of a 2-fold convolution involving the cdf of the random variable of interest.

Numerical integration methods have also received a good deal of attention and most of these have been developed in parallel with the analytic approximations discussed above. One of the earliest references is [7], where the performance of various standard integration methods is analyzed. It is remarked there that approximating the characteristic function via numerical integration is very challenging due to the oscillatory nature of the term $\mathrm{e}^{\mathrm{i}\omega t}$ and the heavy-tailed nature of the lognormal density [cf. 13]. This fact has been further discussed in several other papers [5, 7, 14]). An obvious approach to deal with the oscillations is to employ complex analytic techniques: besides the paper of Holgate [16] where the saddlepoint methodology is exploited, it seems that Gubner [14] was the first in proposing alternative path contours to reduce the oscillatory behavior of the integrand, as followed up by Tellambura and Seranarte [24] where they proposed specific contours passing through the saddlepoint at a steepest descent rate; this choice has the effect that oscillations are removed in a neighborhood around the saddlepoint. The contours proposed are the constant phase contour plus two additional approximating contours; the first is computed numerically, while the other two are given explicitly. In addition, they also address the heavy-tailed nature of the lognormal density by proposing a transformation which delivers an integrand with lighter tails.

Summary of the paper

A closed form expression of the Laplace transform of the lognormal distribution does not exist. Providing a reliable approximation is a difficult problem since traditional approximation methods fail mainly due to the fact that the lognormal distribution is heavy-tailed and its transforms are not analytic in the origin. We proposed a closed form approximation of the Laplace transform which is obtained via a modified version of the classic asymptotic Laplace's method. The main result is a decomposition of the Laplace transform which delivers a closed form approximation and an expression of the exact error. The last turns to be useful to prove the asymptotic equivalence of the proposed approximation. Moreover, since the error term is given in a probabilistic representation it turns out to be convenient for analysis.

In addition, we constructed a Monte Carlo estimator of the Laplace transform of the lognormal distribution. This estimator is based on the probabilistic representation of the error term obtained via the modified version of the Laplace method. We prove the efficiency of this estimator. In contrast, we illustrated that the Crude and Naïve Monte Carlo implementations can deliver unreliable estimates for large values of the argument.

Finally, we conducted numerical experiments where we compared our proposals against other approximations available in the literature. We found that most approximations are very sensible for different values of σ. The method of Tellambura and Seranarte is one of the most precise; however, it delivers unreliable results for small values of σ and sometimes it fails to converge. The proposal of Barakat can deliver sharp results for small values of σ but fails for large values of the argument.

In contrast, we showed, that our closed-form expression (1.3) delivers approximations which remain precise all over the domain of the transform; in particular, it tends to be more precise for small values of σ. An attractive feature of our proposal is its simple closed-form. Moreover, we showed that our efficient IS Monte Carlo estimator and its numerical scheme counterpart in [4] are the only methods which delivered reliable sharp results for any combination of values of the parameters σ and θ. In particular, these remain sharp in asymptotic regions as these are based on an asymptotic method. In addition, these have simple forms and are easy to implement. Furthermore, these do not have convergence issues. Overall, these proposals seem to be excellent options to approximate the Laplace transform of the lognormal distribution.

Appendix A: Numerical scheme of [4]

We use the procedure *integrate* in the R-package to evaluate the integral in (2.1). Let $w(\theta) = \mathcal{W}(\theta\sigma^2)$. Numerical integration is performed from $-\infty$ to ∞ of the function

$$u \mapsto \exp\{-\theta e^{-w(\theta)}(e^{\tau u} - 1) - (\tau^2 u^2/2 - \tau w(\theta)u)/\sigma^2\},$$

giving the result I_0, and the Laplace transform is

$$\mathcal{L}(\theta) = I_0 \exp\{-\theta e^{-w(\theta)} - w(\theta)^2/(2\sigma^2)\}\tau/\sqrt{2\pi\sigma^2}.$$

The scale τ is equal to $\sigma/\sqrt{1 + w(\theta)}$ (the scale used in (2.1)), when this quantity is less than 10, and equal to $\sqrt{w(\theta)^2 + 2w(\theta) + \sigma^2} - w(\theta)$ otherwise. Finally, in the above integration $e^{\tau u}$ is replaced by e^{700} when $\tau u > 700$.

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References

[1] Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables. Applied mathematics series. Dover Publications (1964)

[2] Aitchison, I., Brown, J.A.C.: The lognormal distribution with special reference to its uses in economics. Cambridge University Press, Cambridge (1957)

[3] Asmussen, S., Glynn, P. W.: *Stochastic simulation: algorithms and analysis.* Springer-Verlag, New York, 2007.

[4] Asmussen, S., Jensen, J.L., Rojas-Nandayapa, L.: Exponential family techniques for the lognormal left tail. Research Report (2014)

[5] Barakat, R.: Sums of independent lognormally distributed random variables. Journal of the Optical Society of America **66**, 211–216 (1976)

[6] Barouch, E., Kaufman, G.M.: On sums of lognormal random variables (1976). Working paper. A. P. Sloan School of Management, MIT.

[7] Beaulieu, N.C., Xie, Q.: An optimal lognormal approximation to lognormal sum distributions. IEEE Transactions on Vehicular Technology **53**(2), 479–489 (2004)

[8] Butler, R.W.: Saddlepoint Approximations with Applications. Cambridge University Press (2007)

[9] Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J., Knuth, D.E.: On the Lambert W function. Advances in Computational Mathematics **5**, 329–359 (1996)

[10] Crow, E.L., Shimizu, K.: Lognormal distributions: theory and applications. Marcel Dekker Inc., New York (1988)

[11] de Bruijn, N.G.: Asymptotic Methods in Analysis. Courier Dover Publications (1970)

[12] Dufresne, D.: Sums of lognormals. In: Actuarial Research Conference Proceedings (2008)

[13] Foss, S., Korshunov, D., Zachary, S.: An introduction to heavy-tailed and subexponential distributions. Springer (2011)

[14] Gubner, J.A.: A new formula for lognormal characteristic functions. IEEE Transactions on Vehicular Technology **55**(5), 1668–1671 (2006)

[15] Heyde, C.: On a property of the lognormal distribution. J. Roy. Statist. Soc. Ser. B **29**, 392–393 (1963)

[16] Holgate, P.: The lognormal characteristic function. Communications in Statistical-Theory and Methods **18**, 4539–4548 (1989)

[17] Jensen, J.L.: Saddlepoint Approximations. Oxford Science Publications, Oxford (1994)

[18] Johnson, N.L., Kotz, S., Balakrishnan, N.: Continuous Univariate Distributions, vol. 1, 2 edn. Wiley, New York (1994)

[19] Leipnik, R.B.: On lognormal random variables: I. the characteristic function. Journal of the Australian Mathematical Society Series B **32**, 327–347 (1991)

[20] Limpert, E., Stahel, W.A., Abbt, M.: Log-normal distributions across the sciences: Keys and clues. BioScience **51**, 341–352 (2001)

[21] Rojas-Nandayapa, L.: Risk probabilities: Asymptotics and simulation. Ph.D. thesis, Aarhus University (2008)

[22] Rossberg, A.G.: Laplace transforms or probability distributions and their inversions are easy on logarithmic scales. Journal of Applied Probabilty **45**, 531–541 (2008)

[23] Small, C.G.: Expansions and Asymptotics for Statistics. Chapman & Hall/CRC (2013)

[24] Tellambura, C., Seranarte, D.: Accurate computation of the MGF of the lognormal distribution and its application to sum of lognormals. IEEE Transactions on Communications **58**, 1568–1577 (2010)